Total number of printed pages–23

3 (Sem–5/CBCS) MAT HE 1/2/3

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

OPTION–B

Paper : MAT–HE–5026

(Mechanics)

OPTION–C

Paper : MAT–HE–5036

(Probability and Statistics)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

1. Answer the following questions as directed: $1 \times 10 = 10$

(a) Which of the following Diophantine equations cannot have integer solutions ?

(i) $33x + 14y = 115$

(ii) $14x + 35y = 93$

(b) State whether the following statement is true ***or*** false :

"If a and b are relatively prime positive integers, then the arithmetic progression a, a + b, a + 2b,... contains infinitely many primes."

(c) For any $a \in Z$ prove that $a \equiv a \pmod{m}$, where m is a fixed integer.

(d) Under what condition the k integers $a_1, a_2, \ldots, a_k$ form a CRS (mod m) ?

(e) Find $\sigma(p)$ where p is a prime number.

(f) Define Euler's phi function.

(g) If $n = 12789$, find $\tau(n)$.

(h) If x is a real number then show that $[x] \le x < [x]+1$, where [ ] represents the greatest integer function.

(i) Calculate the exponent of the highest power of 5 that divides 1000!

(j) When an arithmetic function f is said to be multiplicative ?

2. Answer the following questions : $2 \times 5 = 10$

(a) Show that there is no arithmetic progression a, $a + b$, $a + 2b$,.... that consists solely of prime numbers.

(b) Use properties of congruence to show that 41 divides $2^{20} - 1$.

(c) Let $p > 1$ be a positive integer having the property that $p/a\,b \Rightarrow p/a$ or p/b, then prove that p is a prime.

(d) If a is a positive integer and q is its least positive divisor then show that $q \le \sqrt{a}$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^{n} \mu(k!)$, here μ is the Mobius function.

3. Answer ***any four*** questions : 5×4=20

(a) If $(m, n) = 1$ and $S_1 = \{x_0, x_1, x_2, \ldots, x_{m-1}\}$ is a CRS (mod m) and
$S_2 = \{y_0, y_1, y_2, \ldots, y_{n-1}\}$ is a CRS (mod n) then show that the set
$$S = \{nx_i + my_j : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$$
form a CRS (mod mn).

(b) Find all integers that satisfy simultaneously
$$x \equiv 5 \pmod{18};\ x \equiv -1 \pmod{24};$$
$$x \equiv 17 \pmod{33}$$

(c) If $n \geq 1$ is an integer then show that $\sigma(n)$ is odd if and only if n is a perfect square or twice a perfect square.

(d) If $a_1, a_2, \ldots, a_k$ form a RRS (mod m) ie. Reduced Residue System modulo m then show that $k = \phi(m)$.

(e) If x and y be real numbers then show that $[x+y]=[x]+[y]$ and $[-x-y]=[-x]+[-y]$ if and only if one of x or y is an integer.

(f) For $n>2$, show that $\phi(n)$ is an even integer. Here, ϕ is the Euler phi function.

Answer ***either*** (a) **or** (b) from each of the following questions : 10×4=40

4. (a) (i) Show that every positive integer can be expressed as a product of primes. Also show that apart from the order in which prime factors occur in the product, they are unique. 3+4=7

(ii) If k integers $a_1, a_2, \ldots, a_k$ form a CRS (mod m), then show that $m=k$. 3

(b) (i) Show that any natural number greater than one must have a prime factor. 5

(ii) Prove that if all the $n > 2$ terms of the arithmetic progression $p, p+d, p+2d, \ldots, p+(n-1)d$ are prime numbers, then the common difference d is divisible by every prime $q < n$. 5

5. (a) State and prove Wilson's theorem. Is the converse also true ? Justify your answer. 1+6+3=10

(b) Let a and $m > 0$ be integers such that $(a, m) = 1$, then show that $a^{\phi(m)} \equiv 1 \pmod{m}$, here ϕ is the Euler's phi function. Deduce from it the Fermat's Little theorem. Also find the last two digits of 3^{1000}. 5+2+3=10

6. (a) For each positive integer $n \geq 1$, show that

$$\phi(n) = \sum_{d/n} \mu(d)\frac{n}{d} = n\prod_{p/n}\left(1 - \frac{1}{p}\right)$$

(b) (i) If f and g are two arithmetic functions, then show that the following conditions (A) and (B) are equivalent 7

(A) $f(n) = \sum_{d/n} g(d)$

(B) $g(n) = \sum_{d/n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d/n} \mu\left(\frac{n}{d}\right) f(d)$

(ii) If f is a multiplicative arithmetic function, then show that

$g_1(n) = \sum_{d/n} f(d)$ and

$g_2(n) = \sum_{d/n} \mu(d) f(d)$ are both

multiplicative arithmetic functions. 3

7. (a) State and prove Chinese Remainder theorem. 2+8=10

(b) (i) For $n > 1$, show that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n\phi(n)$. 5

(ii) If $n \geq 1$ is an integer then show that $\prod_{d/n} d = n^{\frac{\tau(n)}{2}}$. Is $\prod_{d/n} d$ an integer when $\tau(n)$ is odd ? Justify. 5

OPTION–B

Paper : MAT–HE–5026

(Mechanics)

1. Answer the following questions : 1×10=10

   (a) What is the resultant of two equal forces acting at an angle 120° ?

   (b) State Lami's theorem.

   (c) State the principle of conservation of linear momentum.

   (d) When two parallel forces cannot be compounded into a single resultant force ?

   (e) Define impulsive force with an example.

   (f) State a necessary and sufficient condition for a system of coplanar forces acting on a rigid body to maintain equilibrium.

   (g) Define amplitude and frequency of a simple harmonic motion (SHM).

   (h) Write down the relation between the angle of friction and co-efficient of friction.

(i) State Newton's that law of motion which defines force as the agent of motion change.

(j) What is the graphical representation of the moment of a force?

2. Answer the following questions : $2 \times 5 = 10$

(a) Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.

(b) Find the position of centre of gravity (C.G) of a uniform semicircular arc of radius a.

(c) Prove that earth's gravitational field is a conservative force field.

(d) Two men have to carry a block of stone of weight $70kg$ on a light plank. How must the block be placed so that one of the men should bear a weight of $10kg$ more than the other?

(e) Prove that the change in kinetic energy of a body is equal to the work done by the acting force.

3. Answer the following questions : ***(any four)*** $5\times4=20$

(a) Two forces P and Q acting on a particle at an angle α have a resultant $(2k+1)\sqrt{P^2+Q^2}$. When they act at an angle $90°-\alpha$, the resultant becomes $(2k-1)\sqrt{P^2+Q^2}$, prove that

$$\tan\alpha = \frac{k-1}{k+1}.$$

(b) If the two like parallel forces P and Q acting on a rigid body at A and B be interchanged in position, then show that the point of application of the resultant will be displaced along $\overrightarrow{AB}$ through a distance d where

$$d = \frac{P-Q}{P+Q}.AB \quad (P>Q).$$

(c) Forces of magnitudes 1, 2, 3, 4, $2\sqrt{2}$ act respectively along the sides $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$ and the diagonal $\overrightarrow{AC}$ of the square $ABCD$. Show that their resultant is a couple, and find its moment.

(d) A particle moves towards a centre of attraction starting from rest at a distance a from the centre. If its velocity when at any distance x from the centre vary as $\sqrt{\frac{a^2 - x^2}{x^2}}$, find the law of force.

(e) An elastic string without weight, of which the unstretched length is l and the modulus of elasticity is the weight of n ozs, is suspended by one end, and a mass of m ozs. is attached to the other; show that the time of a vertical oscillation is $2\pi\sqrt{\frac{ml}{ng}}$.

(f) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda - log(1+\lambda)]$, where V is the terminal velocity of the particle and λV is the initial vertical velocity.

4. Answer the following questions : ***(any four)*** 10×4=40

(a) (i) Forces P, Q, R acting along $\overrightarrow{IA}, \overrightarrow{IB}, \overrightarrow{IC}$, where I is the in-centre of the triangle ABC, are in equilibrium. Show that 4

$$P : Q : R = Cos\frac{1}{2}A : Cos\frac{1}{2}B : Cos\frac{1}{2}C$$

(ii) Forces L, M, N act along the sides of the triangle formed by the lines $x + y - 1 = 0$, $x - y + 1 = 0$, $y = 2$. Find the magnitude and the line of action of the resultant. 6

(b) A body is resting on a rough inclined plane of inclination α to the horizon, the angle of friction being $\lambda (\alpha > \lambda)$. If the body is acted on by a force P, then find the magnitude of P when

(i) the body is just on the point of slipping down.

(ii) the body is just on the point of sliding up.

(c) (i) Find the C.G of the area of the cardioid $r = a(1 + cos\theta)$ 5

(ii) Find the C.G. of the solid formed by the revolution of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its minor axis. 5

(d) (i) Three forces P, Q, R act in the same sense along the sides $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ of a triangle ABC. Show that, if their resultant passes through the centroid, then

$$P \, Cosec \, A + Q \, Cosec \, B + R \, Cosec \, C = 0$$

5

(ii) Forces P, Q, R, S act along the sides $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$ of the cyclic quadrilateral $ABCD$, taken in order, where A and B are the extremities of a diameter. If they are in equilibrium, then prove that

$$R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}$$ 5

(e) The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$. Find the path. Also show that the accelerations along and perpendicular to the radius vector are $\lambda^2 r - \frac{\mu^2\theta^2}{r}$ and $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$.

(f) A particle moves in a straight line OA with an acceleration which is always directed towards O and varies inversely as the square of its distance from O. If initially the particle were at rest at A, show that the time taken by it to arrive at the origin is $\dfrac{\pi a^{3/2}}{2\sqrt{2\mu}}$.

(g) Show that the accelerations along the tangent and the normal to the path of a particle are $\dfrac{d^2s}{dt^2}\left(=v\dfrac{dv}{ds}\right)$ and $\dfrac{v^2}{\rho}$, where ρ is the radius of curvature of the curve at the point considered.

(h) A particle falls under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. Discuss the motion, if the particle starts from rest.

OPTION–C

Paper : MAT–HE–5036

(Probability and Statistics)

1. Answer the following questions : $1 \times 10 = 10$

   (a) If A and B are mutually exclusive then find $P(A \cap B)$ and $P(A \cup B)$.

   (b) Define probability mass function for discrete random variable.

   (c) If $P(x) = \frac{x}{15}, \ x = 1$

   0 , elsewhere

   Find $P\{x = 1 \ or \ x = 2\}$

   (d) If X_1 and X_2 are independent random variables then what will be the modified statement of

   $$V(X_1 + X_2) = V(X_1) + V(X_2) + 2cov(X_1, X_2)$$

   (e) If a non-negative real valued function f is the probability density function of some continuous random variable, then what is the value of $\int_{-\alpha}^{\alpha} f(x)dx$?

(f) Name the discrete distribution for which mean and variance have the same value. What is the value?

(g) What is meant by mathematical expectation of a random variable?

(h) Under what condition the binomial distribution becomes the normal distribution.

(i) Write the equation of line of regression of y on x.

(j) State weak law of large number.

2. Answer the following questions : $2\times5=10$

(a) If the events A and B are independent of A and B separately, is it necessary that they are independent of $A \cap B$? Justify.

(b) Let X be a random variable with the following probability distribution :

$$\begin{array}{llll} x: & -3 & 6 & 3 \\ P(X=x): & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{array}$$

Find $E\left(X^2\right)$

(c) State *two* properties of Poisson distribution.

(d) If X is a continuous random variable whose probability density function is given by

$$f(x) = c(4x - 2x^2), 0 < x < 2$$
$$= 0, \text{ otherwise}$$

then find the value of c.

(e) If X is a random variable, then prove that $Var(ax + b) = a^2 Var(X)$ where a and b are constants.

3. Answer ***any four*** parts from the following : 5×4=20

(a) A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second to give 4 black balls if the balls are not replaced before the second draw.

(b) The probability density function of a two dimensional random variable (X, Y) is given by

$$f(x, y) = x + y, 0 < x + y < 1$$
$$0 \quad , \text{ elsewhere}$$

Evaluate $P\left(X < \frac{1}{2}, Y > \frac{1}{4}\right)$

(c) A die is tossed twice. Getting 'a number greater than 4' is considered a success. Find the mean and variance of the probability distribution of the number of success.

(d) The joint density function of two random variables X and Y is given by

$$f(x, y) = \frac{xy}{96}, 0 < x < 4, 1 < y < 5$$
$$0 \text{ , otherwise}$$

Find

(i) $E(X)$

(ii) $E(Y)$

(iii) $E(2X + 3Y)$

(e) For any two independent random variable X and Y, for which $E(X)$ and $E(Y)$ exists, show that

$$E(XY) = E(X)E(Y)$$

(f) With usual notation for a binomial variate X, given that $9p(X = 4) = p(X = 2)$ when $n = 6$

Find the value of p and q.

4. Answer ***any four*** parts from the following : 10×4=40

(a) (i) If A and B are any two events and are not disjoint then show that $P(A \cup B) = P(A) + P(B) - P(A \cup B)$. Hence find $P(A \cup B \cup C)$.

(ii) From a bag containing 4 white and 6 red balls, three balls are drawn at random. Find the expected number of white balls drawn.

(b) (i) The joint density function of x and y is given by

$$f(x, y) = 2e^{-x}e^{-2y}, 0 < x < \alpha, 0 < y < \alpha$$
$$0 \quad , \text{ otherwise}$$

compute $P(X > 1, Y < 1), P(X < Y)$ and $P(X < a)$.

(ii) If X is a random Poisson variate with parameter m, then show that

$$p(X \geq n) - p(X \geq n+1) = \frac{e^{-m}m^n}{L^n}$$

(c) (i) If X is a binomial variate then prove that

$$Cov\left(\frac{X}{n}, \frac{n-X}{n}\right) = -\frac{pq}{n}$$

(ii) Show that normal distribution may be regarded as a limiting case of Poisson's distribution on the parameter $m \to \infty$.

(d) *(i)* Prove that the moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating function.

(ii) Define moments and moment generating function of a random variable X. If $M(t)$ is the moment generating function of a random variable X about the origin, show that the moment μ_r' is given by

$$\mu_r' = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

(e) *(i)* If $U = \frac{X-a}{h}$, $V = \frac{Y-b}{k}$ where a, b, h, k are constants, $h > 0$, $k > 0$ then show that $r(X, Y) = r(U, V)$.

(r represents the correlation co-efficient)

(ii) The two regression equations of the variables x and y are

$$x = 19.13 - 0.87y$$

$$y = 11.64 - 0.50x$$

Find (l) mean of x's

(m) mean of y's

(n) correlation co-efficients between x and y.

(f) (i) Find the mean and variance of a Binomial distribution.

(ii) If X is a random variable with mean μ and variance σ^2 then for any positive number k, prove that

$$p\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

(g) (i) A function $f(x)$ of x is defined as follows :

$$\begin{aligned} f(x) &= 0 && \text{for } x < 2 \\ &= \frac{1}{18}(3 + 2x) && \text{for } 2 \leq x \leq 4 \\ &= 0 && \text{for } x > 4 \end{aligned}$$

Show that it is a density function. Also find the probability that a variate with this density will lie in the interval $2 \leq x \leq 3$.

(ii) Two random variables X and Y have the following joint probability distribution function.

$$f(x, y) = 2 - x - y, \; 0 \le x \le 1, \; 0 \le y \le 1$$
$$= 0 \qquad , \qquad \text{otherwise}$$

Find (l) marginal density function

(m) $E(X)$ and $E(Y)$

(n) conditional density function

(h) (i) Show that Poisson distribution is a limiting case of the Negative Binomial Distribution.

(ii) Let the random variable X_i assume values i and $-i$ with equal probabilities. Show that the law of large number cannot be applied to the independent variables X_1, X_2, X_3, ...X_n.
