2026

MATHEMATICS

(Major)

Paper : MAT0600304

(Metric Space)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×8=8

(a) Describe an open ball in the discrete metric space (X, d).

(b) Define Euclidean metric on $\mathbb{R}^n$.

(c) Find the derived set of

$$F = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$$

(d) Let F_1 and F_2 be two subsets of a metric space (X, d). Then

(i) $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$

(ii) $(F_1 \cap F_2)' = F_1' \cap F_2'$

(iii) $(A \cup B)^{\circ} = A^{\circ} \cup B^{\circ}$

(iv) $(F_1 \cup F_2)' = F_1' \cup F_2'$

(Choose the correct option)

(e) "Cauchy sequences are mapped into Cauchy sequences by a continuous function defined from a metric space to another metric space."

(Write True or False)

(f) The intersection of an infinite number of open sets need not be open. Justify your answer.

(g) Which of the following statements is not true?

(i) Discrete metric space is a complete metric space.

(ii) In a metric space, any intersection of closed sets is closed.

(iii) A Cauchy sequence in a metric space is convergent in it.

(iv) In the discrete metric space, each subset is open.

(Choose the correct option)

(h) Consider the following statements :

I. In a connected metric space (X, d), there exists a proper subset of X that is both open and closed in X.

II. The continuous image of a connected metric space is connected.

In the light of the above statements

(i) both I and II are true

(ii) both I and II are false

(iii) I is true but II is false

(iv) I is false but II is true

(Choose the correct option)

2. Answer any *six* of the following questions : $2 \times 6 = 12$

(a) Prove that a convergent sequence in a metric space is a Cauchy sequence.

(b) Show that each subset of a discrete metric space has no limit point.

(c) Show that the interior of the subset $[0, 1] \subseteq \mathbb{R}$ is the open interval $(0, 1)$.

(d) Let F be a subset of a metric space (X, d). Then prove that F is closed if and only if $F = \overline{F}$.

(e) If F_1 and F_2 are two subsets of a metric space (X, d), then prove that

$$\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$$

(f) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. If $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X, then prove that f is continuous on X.

(g) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous on the metric space $[0, 1]$ with the absolute value metric.

(h) Define homomorphism of metric space and give one example.

(i) Show that the metric spaces (X, d_1) and (X, d_2), where

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)}, \quad \forall\ x, y \in X$$

are equivalent.

(j) Show that the set of rationals $Q \subseteq \mathbb{R}$ with the usual metric induced from $\mathbb{R}$ is not connected.

3. Answer any *four* of the following questions : 5×4=20

(a) Let

$$X = \mathbb{R}^n = \{ x = (x_1, x_2, ..., x_n) : x_i \in \mathbb{R},\ 1 \le i \le n \}$$

be the set of real n-tuples. For $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$, define

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

Verify that d is a metric on $\mathbb{R}^n$.

(b) Define an open ball in a metric space. Prove that in any metric space each open ball is an open set. 1+4=5

(c) Let (X, d) be a metric space and $Y \subseteq X$. Prove that if X is separable, then Y with the induced metric is separable too.

(d) Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.

(e) Define uniformly continuous mapping on a metric space. Show that the function $f:(0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. 1+4=5

(f) Let (X, d) be a metric space and let $x \in X$ and $A \subseteq X$ be non-empty. Prove that $x \in \bar{A}$ if and only if $d(x, A) = 0$.

(g) Let $T: X \to X$ be a contraction of the complete metric space (X, d). Then prove that T has a unique fixed point.

(h) Let (X, d) be a metric space and let $\{Y_\lambda : \lambda \in \Lambda\}$ be a family of connected sets in (X, d) having a non-empty intersection. Prove that $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ is connected.

4. Answer any *two* of the following questions : 10×2=20

(a) *(i)* Prove that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, is a complete metric space. 6

(ii) Let F be a subset of the metric space (X, d). Prove that F' is a closed subset of (X, d). 4

(b) *(i)* Let (X, d) be a metric space and $F \subseteq X$. Then prove that F is closed in X if and only if F^c is open in X. 5

(ii) Let (X, d) be a metric space and Y be a subspace of X. Let $Z \subseteq Y$. Prove that Z is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$. 5

(c) *(i)* If A is a subset of the metric space (X, d), then prove that $d(A) = d(\bar{A})$. 4

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f: A \to Y$ is continuous at $a \in A$ if and only if, whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(d) *(i)* Let (X, d) be a metric space and $F \subseteq X$. Then prove that x_0 is a limit point of F if and only if it is possible to select from the set F a sequence of distinct points $x_1, x_2, \ldots, x_n$ such that

$$\lim_{n} d(x_n, x_0) = 0$$ 5

(ii) Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y where (X, d_X) and (Y, d_Y) are two metric spaces. 5

(e) *(i)* Define discrete two-point space. Let (X, d) be a metric space. Prove that the following statements are equivalent : 1+5=6

1. (X, d) is disconnected.
2. There exists a continuous mapping of (X, d) onto the discrete two-element space (X_0, d_0).

(ii) If Y is a connected set in a metric space (X, d), then prove that any set Z such that $Y \subseteq Z \subseteq \overline{Y}$ is connected. 4

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